

# On quasi-Frobenius semigroup algebras

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## Abstract

We define quasi-Frobenius semigroups and find necessary and sufficient conditions under which a semigroup algebra of a 0-cancellative semigroup is quasi-Frobenius.

I. S. Ponizovskii was first who considered quasi-Frobenius semigroup algebras over a field for finite semigroups. In [7] he described the commutative case completely. Here we study another class of semigroups (0-cancellative). As an example we consider so called modifications of finite groups [3, 4, 5].

A semigroup  $S$  is called *0-cancellative* if for all  $a, b, c \in S$  from  $ac = bc \neq 0$  or  $ca = cb \neq 0$  it follows  $a = b$ . **In what follows  $S$  denotes a finite 0-cancellative semigroup.** As the next assertion shows, these semigroups can be called elementary analogously to [7, 6].

Let  $e$  be an identity of  $S$ ,  $H$  the subgroup of invertible elements of  $S$  (if  $S$  does not contain an identity, we set  $H = \emptyset$ ),  $N = S \setminus H$ .

**Lemma 1**  *$N$  is a nilpotent ideal of  $S$ .*

**Proof.** If  $ab = e$  then  $bab = b$ , whence (0-cancellativity)  $ba = e$ . Hence every right inverse element is also left inverse. This implies that  $N$  is an ideal.

Let  $a \in S$ . Since  $S$  is finite,  $a^m = a^n$  where, e. g.,  $m < n$ . It follows from 0-cancellativity that either  $a \in H$  or  $a^m = 0$ . Therefore  $N$  is a nil-semigroup, and since  $|N| < \infty$ ,  $N$  is nilpotent [2]. ■

If  $A$  is a semigroup or a ring we shall denote by  $l_A(B)$  and  $r_A(B)$  the left and right annihilators of a subset  $B \subset A$ , respectively.

Analogously with Ring Theory, we call a semigroup  $S$  *quasi-Frobenius* if  $r_S l_S(R) = R$  for every right ideal  $R$  and  $l_S r_S(L) = L$  for every left ideal  $L$  of  $S$ .

**Lemma 2** *If  $S$  is quasi-Frobenius then  $S$  contains an identity (so  $H \neq \emptyset$ ).*

**Proof.** Suppose that  $S$  does not have an identity. Then  $S$  is nilpotent by Lemma 1. Let  $S^n = 0$ ,  $S^{n-1} \neq 0$ . Then  $0 = l_S r_S(0) = l_S(S) \supset S^{n-1}$  which is impossible. ■

Let  $F$  be a field,  $F_0 S$  the contracted semigroup algebra of  $S$  (i. e. a factor of  $FS$  obtained by gluing the zeroes of  $S$  and  $FS$ ). For  $A \subset S$  we denote by  $F_0 A$  the image of the vector space  $FA$  in  $F_0 S$ .

**Lemma 3** *If  $F_0 S$  is quasi-Frobenius then  $S$  is quasi-Frobenius too.*

**Proof.** First show that

$$r_{F_0 S}(F_0 A) = F_0 r_S(A) \quad (1)$$

$$l_{F_0 S}(F_0 A) = F_0 l_S(A) \quad (2)$$

for any subset  $A \subset S$ . Indeed,

$$r_{F_0 S}(F_0 A) = r_{F_0 S}(A) \supset F_0 r_S(A).$$

Conversely, if  $\sum_{s \in S} \alpha_s s \in r_{F_0 S}(A)$  ( $\alpha_s \in F$ ) then  $\sum_{s \in S} \alpha_s a s = 0$  for  $a \in A$ .

Different non-zero summands in left side of this equality cannot equal one to another because of 0-cancellativity. Hence

$$\sum_{s \in S} \alpha_s s = \sum_{s \in r_S(A)} \alpha_s s \in F_0 r_S(A).$$

The equation (2) is proved analogously.

Now for right ideal  $R \subset S$  we get from (1) and (2):

$$F_0 r_S l_S(R) = r_{F_0 S}[F_0 l_S(R)] = r_{F_0 S} l_{F_0 S}(F_0 R) = F_0 R.$$

Further,  $F_0 A = F_0 B$  implies  $A \cup 0 = B \cup 0$  (here 0 is the zero of  $S$ ). Since  $0 \in R \subset r_S l_S(R)$  we have  $r_S l_S(R) = R$ .

Similarly  $l_S r_S(L) = L$ . ■

Let  $S$  contains an identity (i. e.  $H \neq \emptyset$ ),  $N^n = 0 \neq N^{n-1}$ . Denote  $M(S) = N^{n-1}$ . Evidently  $S \setminus 0$  is a disjoint union of cosets of  $H$ . Since  $N$  is an ideal,  $HM(S) \subset M(S)$  and  $M(S)$  consists of cosets of  $H$  as well.

**Lemma 4** *If  $S$  is quasi-Frobenius then  $M(S) = Ha \cup 0$  for any  $a \in M(S) \setminus 0$ .*

**Proof.** Let  $a \in M(S)$ . Since  $Ha \cup 0$  is a left ideal in  $S$ ,

$$Ha \cup 0 = l_S r_S(Ha \cup 0) = l_S(N) \supset M(S).$$

So  $Ha \cup 0 = M(S)$ . ■

**Lemma 5** *If  $S$  is quasi-Frobenius then for all  $a \in M(S) \setminus 0$  and  $b \in S \setminus 0$  there is an unique element  $x$  such that  $xb = a$ .*

**Proof.** Uniqueness of  $x$  follows from 0-cancellativity.

The assertion is evident for  $b \in H$ .

Let  $b \in N^k \setminus N^{k+1}$ . If  $k = n - 1$  the statement follows from Lemma 4. We use decreasing induction on  $k$ , supposing that our statement holds for bigger  $k$ 's .

If  $Nb = 0$  then  $b \in r_S(N) = r_S l_S(M(S)) = M(S)$ , i.e.  $k = n - 1$ , the case which has already considered. If  $Nb \neq 0$  then  $cb \neq 0$  for some  $c \in N$ . In this case  $cb \in N^{k+1}$ , so accordingly to the assumption of induction  $xcb = a$  for some  $x$ . ■

Now we are able to prove the main result.

**Theorem 1** *For a finite 0-cancellative semigroup  $S$  the following conditions are equivalent:*

- (i)  $S$  is quasi-Frobenius;
- (ii)  $H \neq \emptyset$  and  $M(S)$  is the least non-zero ideal;
- (iii)  $F_0 S$  is Frobenius;
- (iv)  $F_0 S$  is quasi-Frobenius.

**Proof.**  $1 \implies 2$  was already proved (see Lemmas 2 and 4).

$2 \implies 3$ . We use Theorem 61.3 from [1]. Fix some  $a \in M(S) \setminus 0$  and define a linear function on  $F_0 S$ :

$$f\left(\sum_{s \in S \setminus 0} \alpha_s s\right) = \alpha_a.$$

Every element from  $\text{Ker } f$  is of the form  $A = \sum_{s \neq a} \alpha_s s$ . Let  $\alpha_s \neq 0$ . By Lemma 5 there is  $x \in S$  such that  $xs = a$ . Since  $a \neq 0$ ,  $xt \neq a$  for any  $t \neq s$ . Hence

$xA \notin \text{Ker } f$ , i. e.  $\text{Ker } f$  does not contain left (and similarly right) ideals. By Theorem 61.3 [1],  $F_0S$  is Frobenius.

$3 \implies 4$  is evident.

$4 \implies 1$  was already proved (Lemma 3). ■

\* \* \*

A vast variety of examples of finite 0-cancellative semigroups is given by modifications of groups [3, 4, 5]. Remind that a *modification*  $G(*)$  of a group  $G$  is a semigroup on the set  $G^0 = G \cup \{0\}$  with an operation  $*$  such that  $x*y$  is equal either to  $xy$  or to 0, while

$$0 * x = x * 0 = 0 * 0 = 0$$

and the identity of  $G$  is the same for the semigroup  $G(*)$ .

In other words, to obtain a modification, one must erase the contents of some inputs in the multiplication table of  $G$  and insert there zeros so that the new operation would be associative.

It is clear that modifications are 0-cancellative. Denote as above by  $H$  the subgroup of all invertible elements in  $G(*)$ . By Lemma 1 its complement  $N = G(*) \setminus H$  is a two-sided nilpotent ideal if  $G$  is finite. Let  $N^n = 0 \neq N^{n-1} = M$ . We shall describe quasi-Frobenius modifications for  $n \leq 3$ .

If  $n = 1$  the semigroup  $G(*)$  turns into the group  $G$  with an adjoint zero, so its algebra  $F_0G(*) \cong FG$  is always quasi-Frobenius.

Let  $n = 2$ . If  $G(*)$  is quasi-Frobenius then  $N = M = Ha \cup 0$ . Therefore when  $G$  has a subgroup  $H$  of the index 2 we can build a quasi-Frobenius modification  $G(*)$  giving its multiplication by

$$x * y = \begin{cases} xy & \text{if } x \in H \text{ or } y \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $n = 3$ . Fix  $a \in M \setminus 0$ ; then  $M = Ha \cup 0 = aH \cup 0$ . Let  $b \in N \setminus N^2$ . By Lemma 5 for every  $h \in H$  there is unique  $x \in G(*)$  such that  $x * b = ha$ . Therefore for all  $x \in G(*)$ ,  $b \in N \setminus N^2$  from  $xb \in Ha$  it follows  $x * b \neq 0$ . So the operation  $*$  must have the form

$$x * y = \begin{cases} xy & \text{if } x \in H \text{ or } y \in H \text{ or } xy \in Ha, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Hence we have

**Proposition 1** *Let  $G$  be a finite group,  $H$  its proper subgroup having the non-trivial normalizator  $N_G(H) \neq H$ ,  $a \in N_G(H) \setminus H$ . Then a modification  $G(*)$  is given by (3) is quasi-Frobenius.*

**Proof.** We need only to check associativity of  $*$ . This is equivalent to proving the statement

$$(x * y) * z = 0 \iff x * (y * z) = 0.$$

If  $a, b \in N$  then  $a * b \in M(G(*))$ ; so  $x * y * z = 0$  when  $x, y, z \in N$ . Hence it is sufficient to establish associativity only in the case when one out of elements  $x, y, z$  is contained in  $H$ .

Let, e. g.,  $x \in H$ . We have to prove that

$$xy * z = x(y * z). \quad (4)$$

However,  $xy$  and  $y$  belong or do not belong to  $H$  simultaneously; also  $xyz$  and  $yz$  belong or do not belong to  $aH$  simultaneously. Therefore

$$\begin{aligned} xy * z = 0 &\iff xy \notin H \ \& \ z \notin H \ \& \ xyz \notin H \\ &\iff y \notin H \ \& \ z \notin H \ \& \ yz \notin H \iff y * z = 0 \iff x(y * z) = 0 \blacksquare \end{aligned}$$

We conclude this note by a remark. Theorem 1 and the results of [7] are too similar, nevertheless the considered classes of semigroups (commutative and 0-cancellative) are rather different. It would be of interest to find a larger class of semigroups which includes both above-mentioned classes and yields to some analogue of Theorem 1.

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